11.6.2 Change of Measure for a Compound Poisson Process

312707003 黃鈺婷

Let $N(t)$ be a Poisson process with intensity λ, and let $Y_1, Y_2, \ldots$ be a sequence of identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume the random variables $Y_1, Y_2, \ldots$ are independent of one another and also independent of the Poisson process $N(t)$. We define the *compound Poisson process*

$$\text{總共移動幅度 } \underline{Q(t)} = \sum_{i=1}^{N(t)} Y_i. \qquad (11.6.9)$$

Note for future reference that if N jumps at time t, then Q jumps at time t and

$$Q(t)-Q(t-)=\Delta Q(t) = Y_{N(t)}. \qquad (11.6.10)$$

Jumps in Q(t) **are random sizes**
First jump size: Y_1
Second jump size: Y_2

Q(t):value of Q immediately **after** the jump
Q(t-):value of Q immediately **before** the jump

For a Poisson process, the change of measure affects the intensity.
For a compound Poisson process, the change of measure can affect **both** the intensity and the distribution of jump sizes.

Our goal is to change the measure so that the intensity of $N(t)$ and the distribution of the jump sizes $Y_1, Y_2, \ldots$ both change. We first consider the case when the jump-size random variables have a discrete distribution (i.e., each Y_i takes one of finitely many possible nonzero values $y_1, y_2, \ldots, y_M$). Let $p(y_m)$ denote the probability that a jump is of size y_m:

定義M種jump size的discrete distribution

$$p(y_m) = \mathbb{P}\{Y_i = y_m\}, \ m = 1, \ldots, M.$$

This does not depend on i since $Y_1, Y_2, \ldots$ are identically distributed. We assume that $p(y_m) > 0$ for every m and, of course, that $\sum_{m=1}^{M} p(y_m) = 1$.

Let $N_m(t)$ denote the number of jumps in $Q(t)$ of size y_m up to and including time t, so that

$$N(t) = \sum_{m=1}^{M} N_m(t) \text{ and } Q(t) = \sum_{m=1}^{M} y_m N_m(t).$$

N(t):跳的總次數

第m種jump size跳了$N_m(t)$次

According to Corollary 11.3.4, $N_1, \dots, N_M$ are independent Poisson processes and each N_m has intensity $\lambda_m = \lambda p(y_m)$.

Let $\tilde{\lambda}_1, \dots, \tilde{\lambda}_M$ be given positive numbers, and set

$$Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t} \left(\frac{\tilde{\lambda}_m}{\lambda_m} \right)^{N_m(t)} \text{ and } Z(t) = \prod_{m=1}^{M} Z_m(t). \qquad (11.6.11)$$

進行測度轉換要用到的Z

$$\widetilde{\mathbb{P}}(A) = \int_A Z(T)\, dP \text{ for all } Z \in \mathcal{F}.$$

$$Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t} \left(\frac{\tilde{\lambda}_m}{\lambda_m} \right)^{N_m(t)} \quad \text{and} \quad Z(t) = \prod_{m=1}^{M} Z_m(t). \qquad (11.6.11)$$

Lemma 11.6.4. *The process $Z(t)$ of (11.6.11) is a martingale. In particular, $\mathbb{E}Z(t) = 1$ for all t.*

PROOF: From Lemma 11.6.1, we have

$$dZ_m(t) = \frac{\tilde{\lambda}_m - \lambda_m}{\lambda_m} Z_m(t-)\, dM_m(t), \qquad (11.6.12)$$

where

$$M_m(t) = N_m(t) - \lambda_m\, dt.$$

$$Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t} \left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)} \quad \text{and} \quad Z(t) = \prod_{m=1}^{M} Z_m(t). \qquad (11.6.11)$$

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\,dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))\,dX^c(s)\,dX^c(s) + \sum_{0<s\le t}[f(X(s)) - f(X(s-))]. \qquad (11.5.4)$$

$f(x)=e^x$, $Z_m(t)=f(X(t))$, $X(t)=\underbrace{(\lambda_m-\tilde{\lambda}_m)t}_{X^c(t)}+\underbrace{N_m(t)\log\left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)}_{J(t)}$

$f'(X(t))=f(X(t))=Z_m(t)$

$dX^c(t)=(\lambda_m-\tilde{\lambda}_m)dt$

By 11.5.4, $Z_m(t)=f(X(t))=f(X(0))+(\lambda_m-\tilde{\lambda}_m)\int_0^t f'(X(u))du+\sum_{0<u\le t}[f(X(u))-f(X(u-))]$

$$=Z_m(0)+(\lambda_m-\tilde{\lambda}_m)\int_0^t Z_m(u)du+\sum_{0<u\le t}[Z_m(u)-Z_m(u-)]$$

① jump at time u: $Z_m(u)=\left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)Z_m(u-) \Rightarrow Z_m(u)-Z_m(u-)=\left(\frac{\tilde{\lambda}_m-\lambda_m}{\lambda_m}\right)Z_m(u-)$

② no jump at time u: $Z_m(u)-Z_m(u-)=0 \Rightarrow Z_m(u)-Z_m(u-)=\left(\frac{\tilde{\lambda}_m-\lambda_m}{\lambda_m}\right)Z_m(u-)\underbrace{\Delta N_m(u)}_{=0 \text{ if no jump}}$

$$\therefore \sum_{0<u\le t}[Z_m(u)-Z_m(u-)]=\sum_{0<u\le t}\left(\frac{\tilde{\lambda}_m-\lambda_m}{\lambda_m}\right)Z_m(u-)\Delta N_m(u)=\frac{\tilde{\lambda}_m-\lambda_m}{\lambda_m}\int_0^t Z_m(u-)dN_m(u)$$

$$\therefore Z_m(t)=Z_m(0)+(\lambda_m-\tilde{\lambda}_m)\int_0^t Z_m(u-)du+\frac{\tilde{\lambda}_m-\lambda_m}{\lambda_m}\int_0^t Z_m(u-)dN_m(u)$$

$$=Z_m(0)-\frac{\tilde{\lambda}_m-\lambda_m}{\lambda_m}\int_0^t Z_m(u-)\lambda_m du+\frac{\tilde{\lambda}_m-\lambda_m}{\lambda_m}\int_0^t Z_m(u-)dN_m(u)$$

$$=Z_m(0)+\frac{\tilde{\lambda}_m-\lambda_m}{\lambda_m}\int_0^t Z_m(u-)dM_m(u)$$

$M_m(t)=N_m(t)-\lambda_m dt \Rightarrow dN_m(t)=dN_m(t)-\lambda_m dt$

$$\Rightarrow dZ_m(t)=\frac{\tilde{\lambda}_m-\lambda_m}{\lambda_m}Z_m(t-)dM_m(t)$$

$$Z_m(t) = Z_m(0) + \frac{\tilde{\lambda}_m - \lambda_m}{\lambda_m} \int_0^t Z_m(u-)\, dM_m(t)$$

Because the integrand in (11.6.12) is left-continuous and the compensated Poisson process is a martingale, the process Z_m is a martingale (Theorem 11.4.5).

Theorem 11.4.5. *Assume that the jump process* $X(s)$ *of (11.4.1)–(11.4.3) is a martingale, the integrand* $\Phi(s)$ *is* left-continuous *and adapted, and*

$$\mathbb{E} \int_0^t \Gamma^2(s)\Phi^2(s)\, ds < \infty \text{ for all } t \geq 0.$$

Then the stochastic integral $\int_0^t \Phi(s)\, dX(s)$ *is also a martingale.*

$$\begin{cases} Z_m(u-) \text{ is left continuous} \\ M_m(t) \text{ is martingale} \end{cases} \rightarrow Z_m \text{ is martingale}$$

$\because Z_m(t)$ is martingale, $Z_m(0) = 1$

$\therefore EZ_m(t) = 1$

Corollary 11.5.5 (Itô's product rule for jump processes). *Let $X_1(t)$ and $X_2(t)$ be jump processes. Then*

$$X_1(t)X_2(t) = X_1(0)X_2(0) + \int_0^t X_2(s)\,dX_1^c(s) + \int_0^t X_1(s)\,dX_2^c(s) + [X_1^c, X_2^c](t) + \sum_{0<s\le t}[X_1(s)X_2(s) - X_1(s-)X_2(s-)]$$
$$= X_1(0)X_2(0) + \int_0^t X_2(s-)\,dX_1(s) + \int_0^t X_1(s-)\,dX_2(s) + [X_1, X_2](t). \tag{11.5.11}$$

For $m \neq n$, the Poisson processes N_m and N_n have no simultaneous jumps, and hence $[Z_m, Z_n] = 0$. Itô's product rule (Corollary 11.5.5) implies that

$$d\big(Z_1(t)Z_2(t)\big) = Z_2(t-)\,dZ_1(t) + Z_1(t-)\,dZ_2(t). \tag{11.6.13}$$

Because both Z_1 and Z_2 are martingales and the integrands in (11.6.13) are left-continuous, the process Z_1Z_2 is a martingale. Because Z_1Z_2 has no jumps simultaneous with the jumps of Z_3, Itô's product rule further implies

$$d\big(Z_1(t)Z_2(t)Z_3(t)\big) = Z_3(t-)\,d\big(Z_1(t)Z_2(t)\big) + \big(Z_1(t-)Z_2(t-)\big)\,dZ_3(t).$$

martingale (By Thm 11.4.5) left-continuous martingale $=0$

$$Z_1(t)Z_2(t) = Z_1(0)Z_2(0) + \int_0^t Z_2(u-)\,dZ_1(u) + \int_0^t Z_1(u-)\,dZ_2(u) + [Z_m, Z_n](t)$$

$$\Rightarrow dZ_1(t)Z_2(t) = Z_2(t-)\,dZ_1(t) + Z_1(t-)\,dZ_2(t)$$

Once again, the integrators are martingales and the integrands are left-continuous. Therefore, $Z_1 Z_2 Z_3$ is a martingale. Continuing this process, we eventually conclude that $Z(t) = Z_1(t) Z_2(t) \cdots Z_m(t)$ is a martingale. $\square$

Fix $T > 0$. Because $Z(T) > 0$ almost surely and $\mathbb{E} Z(T) = 1$, we can use $Z(T)$ to change the measure, defining

$$\widetilde{\mathbb{P}}(A) = \int_A Z(T)\, dP \text{ for all } Z \in \mathcal{F}.$$

Theorem 11.6.5 (Change of compound Poisson intensity and jump distribution for finitely many jump sizes). *Under* $\widetilde{\mathbb{P}}$, $Q(t)$ *is a compound Poisson process with intensity* $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$, *and* $Y_1, Y_2, \ldots$ *are independent, identically distributed random variables with*

$$\widetilde{\mathbb{P}}\{Y_i = y_m\} = \tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}. \tag{11.6.14}$$

KEY STEP IN PROOF: We use the independence of $N_1, \ldots, N_M$ under $\mathbb{P}$ to compute the moment-generating function of $Q(t)$ under $\widetilde{\mathbb{P}}$. For $0 \le t \le T$, Lemma 5.2.1 and the moment-generating function formula (11.3.4) imply

$$
\begin{aligned}
\widetilde{\mathbb{E}}\left[e^{uQ(t)}\right] &\overset{①}{=} \mathbb{E}\left[e^{uQ(t)} Z(t)\right] \\
&\overset{②}{=} \mathbb{E}\left[\exp\left\{u \sum_{m=1}^{M} y_m N_m(t)\right\} \cdot \prod_{m=1}^{M} e^{(\lambda_m - \tilde{\lambda}_m)t} \left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}\right] \\
&= \prod_{m=1}^{M} \exp\{(\lambda_m - \tilde{\lambda}_m)t\} \cdot \mathbb{E}\left[\exp\left\{\left(uy_m + \log\frac{\tilde{\lambda}_m}{\lambda_m}\right) N_m(t)\right\}\right] \\
&\overset{③}{=} \prod_{m=1}^{M} \exp\{(\lambda_m - \tilde{\lambda}_m)t\} \exp\left\{\lambda_m t \left(e^{uy_m + \log(\tilde{\lambda}_m/\lambda_m)} - 1\right)\right\} \\
&= \prod_{m=1}^{M} \exp\left\{(\lambda_m - \tilde{\lambda}_m)t + \tilde{\lambda}_m t e^{uy_m} - \lambda_m t\right\} \\
&= \prod_{m=1}^{M} \exp\left\{\tilde{\lambda}_m t (e^{uy_m} - 1)\right\} \\
&\overset{④}{=} \prod_{m=1}^{M} \exp\left\{\tilde{\lambda} t \tilde{p}(y_m) e^{uy_m} - \tilde{\lambda}_m t\right\} \\
&\overset{⑤}{=} \exp\left\{\tilde{\lambda} t \left(\sum_{m=1}^{M} \tilde{p}(y_m) e^{uy_m} - 1\right)\right\}.
\end{aligned}
$$

According to (11.3.5), this is the moment-generating function for a compound Poisson process with intensity $\tilde{\lambda}$ and jump-size distribution (11.6.14). □

Lemma 5.2.1. *Let t satisfying $0 \le t \le T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then*

$$\widetilde{\mathbb{E}} Y = \mathbb{E}[Y Z(t)]. \tag{5.2.8}$$

$$\varphi_{N(t)}(u) = \mathbb{E} e^{uN(t)} = \exp\{\lambda t (e^u - 1)\}. \tag{11.3.4}$$

① By Lemma 5.2.1

② $Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t} \left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}$ and $Z(t) = \prod_{m=1}^{M} Z_m(t)$.

$$Q(t) = \sum_{m=1}^{M} y_m N_m(t)$$

③ By 11.3.4

④ $$\widetilde{\mathbb{P}}\{Y_i = y_m\} = \tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}. \tag{11.6.14}$$

$$\Rightarrow \tilde{\lambda}_m = \tilde{P}(y_m) \cdot \tilde{\lambda}$$

⑤ $$\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$$

The Radon-Nikodým derivative process $Z(t)$ of (11.6.11) may be written as

$$Z(t) = \exp\left\{\sum_{m=1}^{M}(\lambda_m - \tilde{\lambda}_m)t\right\} \cdot \prod_{m=1}^{M}\left(\frac{\tilde{\lambda}\tilde{p}(y_m)}{\lambda p(y_m)}\right)^{N_m(t)} = e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)}.$$

This suggests that if $Y_1, Y_2, \ldots$ are not discrete but instead have a common density $f(y)$, then we could change the measure so that $Q(t)$ has intensity $\tilde{\lambda}$ and $Y_1, Y_2, \ldots$ have a different density $\tilde{f}(y)$ by using the Radon-Nikodým derivative process

$$Z(t) = e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}. \tag{11.6.15}$$

This is in fact the case, although the proof, given below, is harder than the one just given for the case of a discrete jump-size distribution.

To avoid division by zero in (11.6.15), we assume that $\tilde{f}(y) = 0$ whenever $f(y) = 0$. This means that if a certain set of jump sizes has probability zero under $\mathbb{P}$, then it will also have probability zero under $\widetilde{\mathbb{P}}$ considered in Theorem 11.6.7 below.

$$Z(t) = \prod_{m=1}^{M} e^{(\lambda_m - \tilde{\lambda}_m)t}\left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}$$

$$= \exp\left\{\sum_{m=1}^{M}(\lambda_m - \tilde{\lambda}_m)t\right\} \cdot \prod_{m=1}^{M}\left(\frac{\tilde{\lambda}\tilde{p}(y_m)}{\lambda p(y_m)}\right)^{N_m(t)}$$

換次數

$$= \exp\{(\lambda-\tilde{\lambda})t\} \cdot \prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)}$$

discrete換 continuous

$$Z(t) = \exp\{(\lambda-\tilde{\lambda})t\} \cdot \prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$$